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Recovering boundary data: The Cauchy Stokes system

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A B S T R A C T

This paper is concerned with the severely ill-posed Cauchy–Stokes problem. We are interested in a data completion problem which is exploited to detect small leaks to control water loss Kim et al. (2008) [1]. This inverse problem is rephrased into an optimization one: An energy-like error functional is introduced. We prove that the optimality condition of the first order is equivalent to solving an interfacial equation which turns out to be a Cauchy-Steklov-Poincaré operator. Numerical trials highlight the efficiency of the method.

Contents

1. Introduction

Let $\Omega \subset \mathbf{R}^d$, $d = 2, 3$, be a bounded domain with a smooth boundary $\Gamma = \partial\Omega$. We assume that Γ is partitioned into two parts Γ_a and Γ_i having both non-vanishing measure.

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In this work we are interested in a data completion problem related to the Stokes system. It consists in recovering the data on the incomplete (inaccessible,...) boundary Γ_i from the over-specified data on the accessible boundary Γ_a. As application of this problem is the leaks detection which can be useful for the losses of water [1].

Assume a given velocity U and a force F on Γ_a, the data completion problem for the Stokes operator can be formulated as a Cauchy problem type: *find the velocity field u and the pressure p solution to*

$$\begin{cases} -\nu\Delta u + \nabla p = 0 & \text{in } \Omega, \\ \nabla.u = 0 & \text{in } \Omega, \\ u = U & \text{on } \Gamma_a, \\ \sigma(u).n = F & \text{on } \Gamma_a, \end{cases} \tag{1}$$

where ν is the fluid kinematic viscosity and σ is the stress tensor $\sigma(u) = 2\nu D(u) - pI$, $D(u)$ denotes the deformation tensor $D(u) = 1/2(\nabla u + \nabla u^T)$, I is the $d \times d$ identity matrix and n is the unit outward normal vector. This problem is known since Hadamard to be ill-posed in the sense that the dependence of (u,p) on the data (U,F) is not continuous. In order to reconstruct the unknown boundary data $u_{|\Gamma_i}$ and $\sigma(u).n_{|\Gamma_i}$ on Γ_i we will use the Steklov–Poincaré operator (see [2] or [3] or [4] for the Laplace equation). The inverse problem is formulated as an optimization one, the first optimality condition gives rise to an interfacial equation involving the Dirichlet-to-Neumann operator. There is very little litterature dealing with the Stokes–Cauchy problems. We would like to mention the work [5] where the data recovering process reads as a least square tracking of the given data. We will also refer to the alternating iterative algorithm in [6,7] for elliptic equations and in [8–12] for the stationary Stokes system. This paper is outlined as follows: The next Section is devoted to the formulation of the Cauchy problem for the Stokes system. The compatibility data notion is discussed. We recall that the set of compatible data is dense on $H^{1/2}(\Gamma_a)^d \times H^{-1/2}(\Gamma_a)^d$. An energy-like error functional is introduced in the context of the ill-posed problem of recovering boundary data. In Section 2.1, the data completion problem is formulated as an optimization one. In Section 2.2, the first order optimality condition is rephrased in terms of an interfacial problem using the Steklov–Poincaré operator [13,14]. The numerical procedure for solving the Stokes–Cauchy problem is described in Section 3. The Kozlov–Maz'ya–Fomin algorithm (the KMF algorithm) is adapted for the Stokes system in Section 4. Section 5 is devoted to some numerical illustrations to compare the proposed method for the data recovering problem with the KMF algorithm. The closing section is devoted to comments.

2. Formulation of the problem

Let us consider the above Cauchy problem (1). Assume that the data (U,F) are "compatible", i.e. that is this pair is indeed the trace and stress tensors of a unique function (u,p). Extending the data means finding (V,G) such that:

$$\begin{cases} -\nu\Delta u + \nabla p = 0 & \text{in } \Omega, \\ \nabla.u = 0 & \text{in } \Omega, \\ u = U, \sigma(u).n = F & \text{on } \Gamma_a, \\ u = V, \sigma(u).n = G & \text{on } \Gamma_i. \end{cases} \tag{2}$$

The question is to reconstruct numerically the pair (V,G), on the inaccessible boundary Γ_i. However, all the results stated are also true in the case of less smooth boundaries and when Γ_a and Γ_i have contact points. It is proven in [3,15], for the Laplace equation, that the pairs of compatible data (U,F) are dense in $H^{1/2}(\Gamma_a)^d \times H^{-1/2}(\Gamma_a)^d$. In the following lemma we establish that the same density result can be easily extended for the Stokes Cauchy problem.

Lemma 2.1. *Let (U,F) be a given data.*

1. *For a fixed U in $H^{1/2}(\Gamma_a)^d$, the set of data F for which there exists (u,p) in $H^1(\Omega)^d \times L^2(\Omega)$, satisfying the Cauchy problem (1) is everywhere dense in $H^{-1/2}(\Gamma_a)^d$.*
2. *For a fixed F in $H^{-1/2}(\Gamma_a)^d$, the set of data U for which there exists (u,p) in $H^1(\Omega)^d \times L^2(\Omega)$, satisfying the Cauchy problem (1) is everywhere dense in $H^{1/2}(\Gamma_a)^d$.*

Proof. Let us prove the first assertion, the second one can be obtained by the same arguments. It is sufficient to prove the result for $U = 0$. Let (u,p) be the solution to the problem:

$$\begin{cases} -\nu\Delta u + \nabla p = 0 & \text{in } \Omega, \\ \nabla.u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_a, \\ \sigma(u).n = F & \text{on } \Gamma_a. \end{cases} \tag{3}$$

Assume, now, that the first assertion fails. We denote by R the set of the compatible data F with the Dirichlet condition $U = 0$ on Γ_a,

$$R = \{F \in H^{-1/2}(\Gamma_a)^d, \quad \text{such that } (0,F) \quad \text{is a compatible data}\},$$

and the subspace $\overline{R}$ (closure of R) is a proper subspace of $H^{-1/2}(\Gamma_a)^d$. Thus there exists a non-vanishing continuous linear form $l \in H^{1/2}(\Gamma_a)$ such that:

$$\langle l, F \rangle = 0, \quad \forall F \in R. \tag{4}$$

Consider, now, the mixed well-posed following direct problem:

$$\begin{cases} -\nu\Delta v + \nabla q = 0 & \text{in } \Omega, \\ \nabla . v = 0 & \text{in } \Omega, \\ v = l & \text{on } \Gamma_a, \\ \sigma(v).n = 0 & \text{on } \Gamma_i. \end{cases} \tag{5}$$

Applying the second Green's formula to (v, q) and (u, p) we get

$$\int_{\partial\Omega} v\sigma(u).n - u\sigma(v).n = 0$$

The integral is to be understood in the duality sense. Exploiting the boundary data, we obtain:

$$\int_{\Gamma_i} v\sigma(u).n = 0.$$

Let us consider a field $\chi \in \mathscr{C}^\infty(\Gamma_i)^d$, the well-posed mixed problem

$$\begin{cases} -\nu\Delta \mathscr{Z} + \nabla s = 0 & \text{in } \Omega, \\ \nabla . \mathscr{Z} = 0 & \text{in } \Omega, \\ \mathscr{Z} = 0 & \text{on } \Gamma_a, \\ \sigma(\mathscr{Z}).n = \chi & \text{on } \Gamma_i. \end{cases}$$

has a unique solution in $H^1(\Omega)^d \times L^2(\Omega)$. From (2) one gets:

$$\int_{\Gamma_i} \chi . v = 0 \quad \forall \chi \text{ in } \mathscr{C}^\infty(\Gamma_i)^d,$$

and therefore v satisfies the following homogenous Cauchy problem:

$$\begin{cases} -\nu\Delta v + \nabla q & = 0 \quad \text{in } \Omega, \\ \nabla . v & = 0 \quad \text{in } \Omega, \\ v & = 0 \quad \text{on } \Gamma_i, \\ \sigma(v).n & = 0 \quad \text{on } \Gamma_i. \end{cases}$$

Then $v = 0$, which leads to $l \equiv 0$. □

2.1. Minimization problem

In this section we formulate the previous inverse problem as a minimization one. For each given $(v, g) \in H^{1/2}(\Gamma_i)^d \times H^{-1/2}(\Gamma_i)^d$, we consider two mixed well-posed problems. The first one is called the Dirichlet problem (with Dirichlet condition on Γ_a)

$$(P_D)\begin{cases} -\nu\Delta u_D + \nabla p_D & = 0 & \text{in } \Omega, \\ \nabla . u_D & = 0 & \text{in } \Omega, \\ u_D & = U & \text{on } \Gamma_a, \\ \sigma(u_D).n + \alpha u_D & = g + \alpha v & \text{on } \Gamma_i, \end{cases}$$

where α is a scalar parameter.

The second one is called the Neumann problem (with Neumann condition on Γ_a)

$$(P_N)\begin{cases} -\nu\Delta u_N + \nabla p_N & = 0 & \text{in } \Omega, \\ \nabla . u_N & = 0 & \text{in } \Omega, \\ \sigma(u_N).n & = F & \text{on } \Gamma_a \\ \sigma(u_N).n + \beta u_N & = g + \beta v & \text{on } \Gamma_i, \end{cases}$$

where β is a scalar parameter.

Then, the unknown data (V, G) can be characterized as the solution of the following minimization problem ([3,16]):

$$(V, G) = arg \min_{v,g} E_{\alpha,\beta}(v, g).$$

where $E_{\alpha,\beta}$ is the following energy-like error functional defined on $H^{1/2}(\Gamma_i)^d \times H^{-1/2}(\Gamma_i)^d$ by

$$E_{\alpha,\beta}(v,g) = \frac{1}{2}\int_\Omega \sigma(u_D - u_N) : \nabla(u_D - u_N)\mathrm{dx}. \tag{6}$$

For the problems (P_D) and (P_N), the introduced parameter (α,β) permit us to specify different types of boundary conditions on Γ_i. Throughout the paper we will treat the minimization problem using one of the following conditions:

- The Neumann-Dirichlet case which will be denoted by *ND*. It corresponds to $\alpha = 0$ (i.e. (P_D) with Neumann boundary condition on Γ_i) and $\beta = +\infty$ (i.e. (P_N) with Dirichlet boundary condition on Γ_i).
- The Dirichlet-Dirichlet case which will be denoted by *DD*. It corresponds to $\alpha = \beta = +\infty$. In this case, the first order optimality condition leads to the variational form of the Steklov-Poincaré operator.
- The Neumann-Neumann case which will be denoted by *NN*. It corresponds to $\alpha = \beta = 0$. It describes the so-called dual Steklov-Poincaré operator.

Remark 1. The energy-type error functional has been already introduced for data completion in the framework of Laplace equation [3,17]. The approach followed here solves, as shown in the next section, the interfacial equation rather than the optimization problem.

2.2. The first order optimality condition

We derive here the first order optimality condition for the previous minimization problem. In the case of compatible data we have the following result.

Theorem 2.2. *When (U,F) is a compatible pair, the minimum of $E_{\alpha,\beta}$ is reached when:*

$$\begin{aligned} &u_D = u_N + Const \quad \text{on } \Gamma_i,\\ &\sigma(u_D).n = \sigma(u_N).n \quad \text{on } \Gamma_i. \end{aligned} \tag{7}$$

Proof. We derive the first optimality conditions in the three considered cases.

(i) *The Neumann–Dirichlet case:* The problems (P_D) and (P_N) are considered using, respectively, Neumann and Dirichlet conditions on Γ_i. For each given $(v,g) \in H^{1/2}(\Gamma_i)^d \times H^{-1/2}(\Gamma_i)^d$, we have the following mixed well-posed problems

$$(P_D)\begin{cases} -\nu\Delta u_D^g + \nabla p_D^g &= 0 \quad \text{in } \Omega,\\ \nabla.u_D^g &= 0 \quad \text{in } \Omega,\\ u_D^g &= U \quad \text{on } \Gamma_a,\\ \sigma(u_D^g).n &= g \quad \text{on } \Gamma_i, \end{cases} \quad (P_N)\begin{cases} -\nu\Delta u_N^v + \nabla p_N^v &= 0 \quad \text{in } \Omega,\\ \nabla.u_N^v &= 0 \quad \text{in } \Omega,\\ \sigma(u_N^v).n &= F \quad \text{on } \Gamma_a,\\ u_N^v &= v \quad \text{on } \Gamma_i. \end{cases}$$

Using the Green formula, the partial derivative of $E_{ND} = E_{0,+\infty}$ with respect to v is given by

$$\frac{\partial E_{ND}}{\partial v}(w) = \int_\Omega 2\nu D(u_N^v - u_D^g) : \nabla r_N^w \mathrm{dx} = \int_{\partial\Omega} \sigma(r_N^w).n(u_N^v - u_D^g)\mathrm{ds},$$

where (r_N^w, s_N^w) is the solution to

$$\begin{cases} -\nu\Delta r_N^w + \nabla s_N^w &= 0 \quad \text{in } \Omega\\ \nabla.r_N^w &= 0 \quad \text{in } \Omega\\ \sigma(r_N^w).n &= 0 \quad \text{on } \Gamma_a\\ r_N^w &= w \quad \text{on } \Gamma_i. \end{cases} \tag{8}$$

Since $\sigma(r_N^w).n = 0 \text{on} \Gamma_a$, we get

$$\frac{\partial E_{ND}}{\partial v}(w) = \int_{\Gamma_i} \sigma(r_N^w).n(u_N^v - u_D^g)\mathrm{ds} \quad \forall w \in H^{1/2}(\Gamma_i)^d. \tag{9}$$

In a similar way we derive the partial derivative of E_{ND} with respect to g

$$\frac{\partial E_{ND}}{\partial g}(h) = \int_\Omega 2\nu D(u_D^g - u_N^v) : \nabla r_D^h \mathrm{dx} = \int_{\Gamma_i} \sigma(u_D^g - u_N^v).n r_D^h \mathrm{ds} \quad \forall h \in H^{-1/2}(\Gamma_i)^d, \tag{10}$$

where (r_D^h, s_D^h) is the solution to

$$\begin{cases} -\nu\Delta r_D^h + \nabla s_D^h & = 0 \quad \text{in } \Omega \\ \nabla . r_D^h & = 0 \quad \text{in } \Omega \\ r_D^h & = 0 \quad \text{on } \Gamma_a \\ \sigma(r_D^h).n & = h \quad \text{on } \Gamma_i. \end{cases} \tag{11}$$

Consider the Steklov–Poincaré operator

$$\begin{aligned} S_N : H^{1/2}(\Gamma_i)^d \; &\to \; H^{-1/2}(\Gamma_i)^d \\ w \; &\to \; \sigma(r_N^w).n. \end{aligned} \tag{12}$$

One can observe that the kernel $N(S_N)$ and the range $R(S_N)$ of the operator S_N are defined by

$$N(S_N) = \mathbb{R} \quad \text{and} \quad R(S_N) = H^{1/2}(\Gamma_i)^d.$$

Then, it follows that $S_N : H^{1/2}(\Gamma_i)^d / N(S_N) \to H^{-1/2}(\Gamma_i)^d$ is an isomorphism. Consequently, the Eq. (9) implies the first condition of the Theorem 2.2:

$$u_N - u_D = \text{Const} \quad \text{on } \Gamma_i.$$

For the second condition, we introduce the dual Steklov-Poincaré operator:

$$\begin{aligned} S_D^{-1} \; : \; H^{-1/2}(\Gamma_i)^d \; &\to \; H^{1/2}(\Gamma_i)^d \\ h \; &\to \; r_D^h. \end{aligned} \tag{13}$$

From the fact that (U, F) is a compatible pair one can deduce that S_D^{-1} is an isomorphism. Then the equality

$$\sigma(u_D).n = \sigma(u_N).n \quad \text{on } \Gamma_i,$$

follows immediately from the Eq. (10).

(ii) *The Dirichlet-Dirichlet case:* We consider the problems (P_D) and (P_N) with Dirichlet conditions on Γ_i. Then, we have

$$(P_D)\begin{cases} -\nu\Delta u_D^v + \nabla p_D^v & = 0 \quad \text{in } \Omega \\ \nabla . u_D^v & = 0 \quad \text{in } \Omega \\ u_D^v & = U \quad \text{on } \Gamma_a \\ u_D^v & = v \quad \text{on } \Gamma_i, \end{cases} \qquad (P_N)\begin{cases} -\nu\Delta u_N^v + \nabla p_N^v & = 0 \quad \text{in } \Omega \\ \nabla . u_N^v & = 0 \quad \text{in } \Omega \\ \sigma(u_N^v).n & = F \quad \text{on } \Gamma_a \\ u_N^v & = v \quad \text{on } \Gamma_i, \end{cases}$$

for each $v \in H^{1/2}(\Gamma_i)^d$.

Using the Green formula, we derive

$$E_{+\infty,+\infty}(v) = E_{DD}(v) = 1/2 \int_\Omega \sigma(u_D^v - u_N^v) : \nabla(u_D^v - u_N^v).$$

The partial derivative of E_{DD} with respect to v is given by:

$$\frac{\partial E_{DD}}{\partial v}(h) = \int_\Omega \sigma(u_D^v(v) - u_N^v(v)) : \nabla(r_D^h - r_N^h) \quad \forall h \in H^{1/2}(\Gamma_i)^d,$$

where (r_D^h, s_D^h) and (r_N^h, s_N^h) are respectively the solution to

$$\begin{cases} -\nu\Delta r_D^h + \nabla s_D^h & = 0 \quad \text{in } \Omega \\ \nabla . r_D^h & = 0 \quad \text{in } \Omega \\ r_D^h & = 0 \quad \text{on } \Gamma_a \\ r_D^h & = h \quad \text{on } \Gamma_i, \end{cases} \begin{cases} -\nu\Delta r_N^h + \nabla s_N^h & = 0 \quad \text{in } \Omega \\ \nabla . r_N^h & = 0 \quad \text{in}\Omega \\ \sigma(r_N^h).n & = 0 \quad \text{on } \Gamma_a \\ r_N^h & = h \quad \text{on } \Gamma_i. \end{cases}$$

From the weak formulation of the last problems we obtain:

$$\frac{\partial E_{DD}}{\partial v}(h) = \int_{\partial\Omega} \sigma(u_D^v(v) - u_N^v(v)).n.(r_D^h) + \int_{\partial\Omega} (u_D^v(v) - u_N^v(v)).\sigma(r_N^h).n.$$

Using the fact that $r^h_{D\Gamma_a} = 0$ and $\sigma(r_N^h).n_{\Gamma_a} = 0$, we get

$$\frac{\partial E_{DD}}{\partial v}(h) = \int_{\Gamma_i} \sigma(u_D^v(v) - u_N^v(v)).n.h, \quad \forall h \in H^{1/2}(\Gamma_i)^d.$$

The second condition of the Theorem 2.2 follows immediately from the last equation.

(iii) *The Neumann-Neumann case:* Here we impose Neumann conditions on Γ_i. This case corresponds to the so-called dual Cauchy-Steklov-Poincaré operator. We consider the following mixed boundary value problems:

$$(P_D)\begin{cases} -\nu\Delta u_D^g + \nabla p_D^g &= 0 \quad \text{in } \Omega,\\ \nabla . u_D^g &= 0 \quad \text{in } \Omega,\\ u_D^g &= U \quad \text{on } \Gamma_a,\\ \sigma(u_D^g).n &= g \quad \text{on } \Gamma_i,\end{cases} (P_N)\begin{cases} -\nu\Delta u_N^g + \nabla p_N^g &= 0 \quad \text{in } \Omega,\\ \nabla . u_N^g &= 0 \quad \text{in } \Omega,\\ \sigma(u_N^g).n &= F \quad \text{on } \Gamma_a,\\ \sigma(u_N^g).n &= g \quad \text{on } \Gamma_i.\end{cases}$$

The Theorem 2.2 is fulfilled when $u_D^g = u_N^g + \text{const}$, which can be expressed by: $E_{NN}(g) = \int_\Omega D(u_D^g - u_N^g) : \nabla(u_D^g - u_N^g)$.
The partial derivative of E_{NN} with respect to g can be written as:

$$\frac{\partial E_{NN}}{\partial g}(h) = \int_\Omega D(u_D^g(g) - u_N^g(g)) : \nabla(r_D^h - r_N^h) \quad \forall h \in H^{-1/2}(\Gamma_i)^d,$$

where (r_D^h, s_D^h) and (r_N^h, s_N^h) are respectively the solution to

$$\begin{cases} -\nu\Delta r_D^h + \nabla s_D^h &= 0 \quad \text{in } \Omega\\ \nabla . r_D^h &= 0 \quad \text{in } \Omega\\ r_D^h &= 0 \quad \text{on } \Gamma_a\\ \sigma(r_D^h).n &= h \quad \text{on } \Gamma_i,\end{cases} \begin{cases} -\nu\Delta r_N^h + \nabla s_N^h &= 0 \quad \text{in } \Omega\\ \nabla . r_N^h &= 0 \quad \text{in}\Omega\\ \sigma(r_N^h).n &= 0 \quad \text{on } \Gamma_a\\ \sigma(r_N^h).n &= h \quad \text{on } \Gamma_i.\end{cases}$$

Using Green formula and the fact that $r_D^h = 0$ and $\sigma(r_N^h).n = 0$ on Γ_a, we obtain

$$\frac{\partial E_{NN}}{\partial g}(h) = \int_{\Gamma_i} (u_D^g - u_N^g).h, \quad \forall h \in H^{-1/2}(\Gamma_i)^d$$

which implies the first condition of (7). □

2.3. The interfacial operators

In this section we introduce interfacial operators. For each case, we rephrase the first order optimality condition, described in the previous section, in term of an interfacial operator.

2.3.1. The Neumann-Dirichlet case:

The solutions (u_D^g, p_D^g) and (u_N^ν, p_N^ν) can be decomposed as

$$(u_D^g, p_D^g) = (u_D^0, p_D^0) + (r_D^g, s_D^g) \quad \text{and} \quad (u_N^\nu, p_N^\nu) = (u_N^0, p_N^0) + (r_N^\nu, s_N^\nu).$$

Then, the equalities (7) can be rewritten as

$$\begin{cases} r_N^\nu - r_D^g &= u_D^0 - u_N^0 \quad \text{on } \Gamma_i\\ \sigma(r_N^\nu).n - \sigma(r_D^g).n &= \sigma(u_D^0).n - \sigma(u_N^0).n \quad \text{on } \Gamma_i.\end{cases}$$

Using the definitions of the fields r_N^ν and r_D^g, we deduce the following interfacial system satisfied by (ν, g)

$$\begin{cases} \nu - S_D^{-1}(g) &= u_D^0 - u_N^0 \quad \text{on } \Gamma_i\\ -S_N(\nu) + g &= \sigma(u_N^0).n - \sigma(u_D^0).n \quad \text{on } \Gamma_i,\end{cases}$$

which can be written as:

$$S\begin{pmatrix} \nu \\ g \end{pmatrix} = T,$$

where $T = \begin{pmatrix} u_D^0 - u_N^0 \\ \sigma(u_N^0).n - \sigma(u_D^0).n \end{pmatrix}$ only depends on the data (U, F) and

$$S = \begin{pmatrix} I & -S_D^{-1} \\ -S_N & I \end{pmatrix}.$$

2.3.2. The Dirichlet–Dirichlet case:

We decompose the solutions (u_D^ν, p_D^ν) and (u_N^ν, p_N^ν) as

$$(u_D^\nu, p_D^\nu) = (u_D^0, p_D^0) + (r_D^\nu, s_D^\nu) \quad \text{and} \quad (u_N^\nu, p_N^\nu) = (u_N^0, p_N^0) + (r_N^\nu, s_N^\nu).$$

According to the previous Theorem, when the minimum is reached, we have

$$\begin{cases} u_D^\nu &= u_N^\nu \quad \text{on } \Gamma_i,\\ \sigma(u_N^\nu).n &= \sigma(u_D^\nu).n \quad \text{on } \Gamma_i.\end{cases}$$

The first condition is always fulfilled. The second one reads

$$\sigma(r_D^v).n - \sigma(r_N^v).n = -(\sigma(u_D^0).n - \sigma(u_N^0).n) \quad \text{on } \Gamma_i.$$

This identity amounts to the requirement that v satisfies the Steklov-Poincaré type equation:

$$S(v) = T \quad \text{on } \Gamma_i, \tag{14}$$

where $T = -(\sigma(u_D^0).n - \sigma(u_N^0)).n$, and S is the Stokes–Cauchy–Steklov–Poincaré operator formally defined by

$$S(v) = (S_D - S_N)(v) = \sigma(r_D^v).n - \sigma(r_N^v).n.$$

2.3.3. The Neuman–Neuman case:

In this case the first relation in (7) gives

$$u_D^g = u_N^g, \quad \text{and} \quad S(g) = T \quad \text{on } \Gamma_i,$$

where $T = -(u_D^0 - u_N^0)$, and S is defined by $S(g) = S_D^{-1} - S_N^{-1} = r_D^g - r_N^g$.

3. The numerical procedure

In this section we propose a numerical procedure for the reconstruction of the data on the inaccessible boundary Γ_i. We consider the two cases ND and DD conditions given in the previous section. In each case, we solve the appropriate interfacial problem. We have used an iterative process based on the preconditioned gradient algorithm:

$$X_{k+1} = X_k - \rho P[S(X_k) - T],$$

where P is a preconditioning operator and ρ is a relaxation parameter. The expressions of S and T are described in the previous section. The expressions of P and X_k are given in the Table 1. As a stopping criterion we choose the first k when the energy-functional $E_{\alpha,\beta}$ is less than a given tolerance level ε.

The main steps of the proposed numerical procedure for the Neumann–Dirichlet case are described by the following algorithm. The same algorithm can be adapted for DD condition on Γ_i.

The algorithm: "the Neumann-Dirichlet case".

- Initialization: set $k = 0$ and chosen v_0 and g_0.
- The stopping criteria: $E_{0,+\infty}(v_k, g_k) \leqslant \varepsilon$, where ε is a given tolerance level:
  - solve the problems (P_D) and (P_N) (using $v = v_k$ and $g = g_k$)
  - computation of the gradient: compute w_D^k and w_N^k solutions to the problems (A_D) and (A_N),
  - set $v_{k+1} = v_k - \rho w_N^k$ and $g_{k+1} = g_k - \rho w_D^k$,
  - $k \to k+1$,

where (A_D) and (A_N) are two auxiliary preconditioners problems defined by

$$(A_N)\begin{cases} -\nu\Delta w_N^k + \nabla q_N^k & = 0 \quad \text{in } \Omega \\ \nabla . w_N^k & = 0 \quad \text{in } \Omega \\ w_N^k & = u_D^g - u_N^v \quad \text{on } \Gamma_i \\ \sigma(w_N^k).n & = 0 \quad \text{on } \Gamma_a, \end{cases}$$

$$(A_D)\begin{cases} -\nu\Delta w_D^k + \nabla q_D^k & = 0 \quad \text{in } \Omega \\ \nabla . w_D^k & = 0 \quad \text{in } \Omega \\ \sigma(w_D^k).n & = (\sigma(u_N^v) - \sigma(u_D^g)).n \quad \text{on } \Gamma_i \\ w_D^k & = 0 \quad \text{on } \Gamma_a. \end{cases}$$

Table 1
The expressions of P and X_k.

	ND	DD
P	$\begin{pmatrix} I & 0 \\ 0 & S_D^{-1} \end{pmatrix}$	S_D
X_k	$\begin{pmatrix} v_k \\ g_k \end{pmatrix}$	v_k

4. Comparison with Kozlov–Maz'ya–Fomin's algorithm

The Kozlov–Maz'ya–Fomin's [6] algorithm has been adapted in [8] for Stokes systems. In this approach, the data completion problem is solved on the basis of an alternating iterative procedure, where successive solutions of well-posed mixed boundary value problems for the original equation are computed. The KMF's algorithm can be viewed as an energy-like error functional minimization by an alternating procedure in the g and v directions. Indeed, problem (2) can be split into two well-posed subproblems with mixed boundary conditions.

$$(P_D)\begin{cases} -\nu\Delta u_D^g + \nabla p_D^g & = 0 \quad \text{in } \Omega, \\ \nabla . u_D^g & = 0 \quad \text{in } \Omega, \\ u_D^g & = U \quad \text{on } \Gamma_a, \\ \sigma(u_D^g).n & = g \quad \text{on } \Gamma_i, \end{cases} \quad (P_N)\begin{cases} -\nu\Delta u_N^v + \nabla p_N^v & = 0 \quad \text{in } \Omega, \\ \nabla . u_N^v & = 0 \quad \text{in } \Omega, \\ \sigma(u_N^v).n & = F \quad \text{on } \Gamma_a, \\ u_N^v & = v \quad \text{on } \Gamma_i. \end{cases}$$

As we can notice this problem corresponds to the Neumann–Dirichlet case. Therefore, solving the Cauchy system (2) is achieved when the data completion (G, V) leads to the same field $u_D^g = u_N^v$ in Ω.

The iterative data completion procedure of [6] can be summarized as follows: starting from an initial guess g on Γ_i, this guess is iteratively corrected by solving alternately problems of form (P_D) and (P_N), where at each iteration the appropriate boundary data results from the solution of the previously solved boundary value problems. A sequence of well-posed mixed problems is generated as follows: u^{2j+1} solves (P_N) with v replaced by u^{2j}, while u^{2j+2} solves (P_D) with g replaced by $\sigma(u^{2j+1}).n$.

Reverting to our energy-like error functional, the link with KMF's algorithm is revealed by the following proposition:

Theorem 4.1. *The KMF's algorithm can be interpreted as an alternating-direction minimization method for the energy-like error functional E_{ND}. More precisely:*

- *Step $2j+1$ of KMF algorithm: u^{2j+1} is characterized by:*

  $$u^{2j+1} = u_N^v(v^{2j+1}) \Longleftrightarrow v^{2j+1} = \arg\min E_{ND}(g^{2j}, v) \quad \text{with } g^{2j} = \sigma(u^{2j}).n|_{\Gamma_i}$$

- *Step $2j+2$ of KMF algorithm: u^{2j+2} leads to:*

  $$u^{2j+2} = u_D^g(g^{2j+2}) \Longleftrightarrow g^{2j+2} = \arg\min E(g, v^{2j+1}) \quad \text{with } v^{2j+1} = u^{2j+1}|_{\Gamma_i}$$

Proof. The proof relies on the careful examination of the different steps of the alternating-direction algorithm for the minimization method of $E_{ND}(g, v)$, namely the minimization with respect to one field, the other being kept fixed.

1. Minimizing in the v direction leads to v_o such that:

$$\min E_{ND}(g_0, v), v \in (H^{1/2}\Gamma_i)^d.$$

Where $g_0 = \sigma(u^{2j}).n$. By the first optimality condition, and using an equivalent form of the derivative (9), one obtains:

$$\frac{\partial E_{ND}(g_0, v_0)}{\partial v}.h = 0, \quad \forall h \in H^{1/2}(\Gamma_i)^d = \int_{\Gamma_i} [u_N(v_0) - u_D(g_0)]\sigma(r_N^h).n + \int_{\Gamma_a} [u_N(v_0) - u_D(g_0)]\sigma(r_N^h).n \quad \forall h \in H^{1/2}(\Gamma_i)^d$$

Recalling that r_N^h satisfies (8), one gets:

$$\int_{\Gamma_i} (u_D(g_0) - u_N(v_0)).\sigma(r_N^h).n = 0 \quad \forall h \in H^{1/2}(\Gamma_i)^d.$$

Consider the Poincaré–Steklov operator

$$S_N : (H^{1/2}(\Gamma_i))^d \rightarrow (H^{-1/2}(\Gamma_i))^d$$
$$h \rightarrow \sigma(r_N^h).n$$

Using the same argument as the Neumann–Dirichlet, we gets that:

$$u_N(v_0) = u_D(g_0) \quad \text{on } \Gamma_i.$$

Then v_0, the minimizer of $E_{ND}(g_0, v)$ is associated to $(u_N(v_0), p_N(v_0))$ which turns out to be the (u^{2j+1}, p^{2j+1}) solution of (P_N). The first claim of Proposition 4.1 is then proved.

2. Minimizing in the g direction leads to g_0 such that:

$$\min E_{ND}(g, v_0) g \in H^{-1/2}(\Gamma_i)^d. \tag{15}$$

Once again by the first optimality condition, and using an equivalent form of the derivative (10), one gets:

$$\frac{\partial E_{ND}(g_0, v_0)}{\partial v}.\psi = 0, \quad \forall \psi \in H^{-1/2}(\Gamma_i)^d = \int_{\Gamma_i} [\sigma(u_N(v_0)) - \sigma(u_D(g_0))].nr_D^\psi + \int_{\Gamma_a} [\sigma(u_N(v_0)) - \sigma(u_D(g_0))].nr_D^\psi, \quad \forall \psi \in H^{1/2}(\Gamma_i)^d.$$

Recalling that r_D^ψ satisfies (11), one gets:

$$\int_{\Gamma_i} (\sigma(u_D(g_0)) - \sigma(u_N(v_0))).n.r_D^\psi = 0, \quad \forall \psi \in H^{-1/2}(\Gamma_i)^d.$$

Using the inverse of Poincaré–Steklov operator:

$$\begin{aligned} S_D^{-1} : H^{-1/2}(\Gamma_i)^d &\rightarrow H^{1/2}(\Gamma_i)^d \\ \psi &\rightarrow r_D^\psi \end{aligned}$$

one gets that:

$$\sigma(u_D(g_0)).n = \sigma(u_N(v_0)).n \quad \text{on } \Gamma_i \qquad \square$$

5. Numerical illustration

Let us consider a viscous incompressible fluid that is confined between two concentric circular cylinders of infinite length. We assume that the velocity field and the pressure do not depend on the longitudinal coordinate. We deal therefore with a two dimensional problem defined in a cross section Ω.

The domain Ω, the accessible boundary Γ_a and the inaccessible boundary Γ_i are depicted in the Fig. 1.

Aiming to validate the proposed approach, we consider here the identification of the velocity field and the stress force on the inner circle Γ_i form an over-specified data (U, F) on the outer circle Γ_a. The data (U, F) is generated from analytic solutions. The numerical experiments are performed on a thick annular domain with radii $R_1 = 2$ and $R_2 = 1$. The numerical simulation is run under the *Freefem++* Software environment [18], it is a free software based on the Finite Element Method. The domain Ω is discretized using an uniform mesh with 50 nodes on Γ_i and 100 nodes on Γ_a. The reconstructed data are computed using the iterative algorithm described in Section 3.

We consider two test cases. In the first one, we take a polynomial example (smooth data). In the second one, we use a singular data.

First test: we take a polynomial example, the velocity u and the pressure p are given by:

$$u(x,y) = (4y^3 - x^2, 4x^3 + 2xy - 1), p(x,y) = 24xy - 2x, \forall (x,y) \in \Omega.$$

We have reconstructed the unknown data on Γ_i in the three cases: Neumann-Dirichlet (*ND*), Dirichlet-Dirichlet (*DD*) and KMF's algorithm (KMF). The obtained results of this test are presented in Figs. 2 and 3. In Fig. 2 we plot the reconstructed velocity $v = (v1, v2)$. In Fig. 3 we plot the reconstructed stress tensor $g = (g1, g2)$. Note that the reconstructed fields are in close agreement with the exact ones. To emphasize further the reliability of our numerical procedure, we have reconstructed the velocity and the stress tensor on Γ_i from some noisy data. More precisely, the data U is polluted by a pointwise white noise with an amplitude ranging from 0 to 0.05. We observe that the reconstruction process remains robust for reasonnable levels of noise (less than 5% of noise). We present in Fig. 4 the recovered data is for 5% of noise. We have only plotted the first components v_1 and g_1.

Second test: we consider the example

$$\begin{cases} u(x,y) = \frac{1}{4\pi}\left(\log\frac{1}{\sqrt{(x-a)^2+y^2}} + \frac{(x-a)^2}{(x-a)^2+y^2}, \frac{y(x-a)}{(x-a)^2+y^2}\right) & \forall (x,y) \in \Omega, \\ p(x,y) = \frac{1}{2\pi}\frac{x-a}{(x-a)^2+y^2} \quad \forall (x,y) \in \Omega. \end{cases}$$

As one can remark, the second test involves a singularity in the vicinity of the inner boundary. It is noticed that the algorithm has difficulties to converge in the presence of singularities, we choose $\varepsilon = 10^{-5}$ where the reconstruction is very satisfactory. For the first case the source is in the vicinity of Γ_i with $a = 0.4$ whereas in the second case it is in the vicinity of Γ_a with

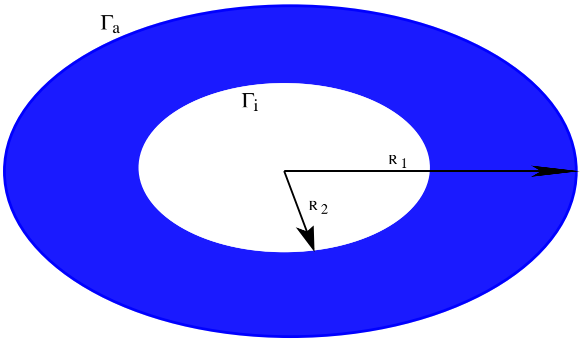
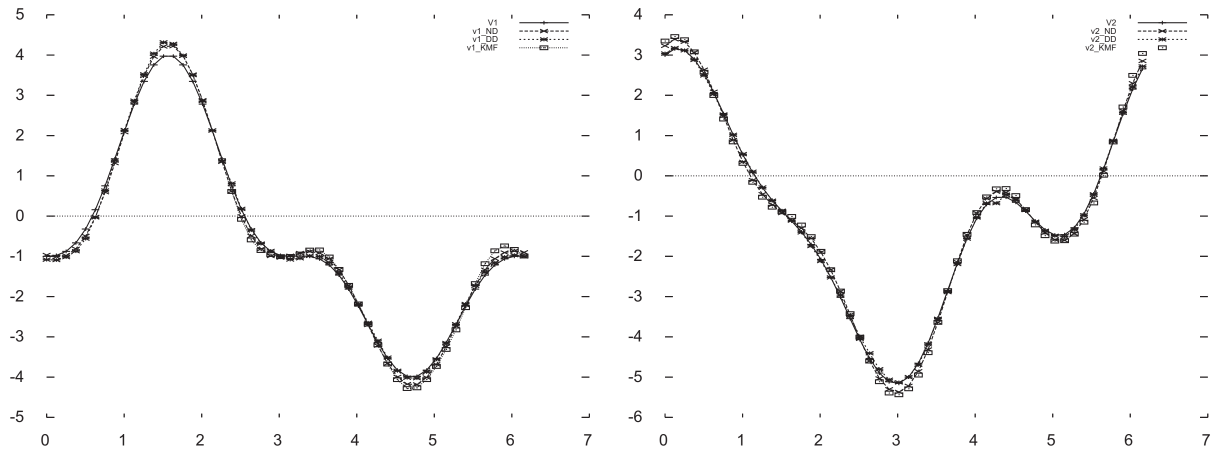
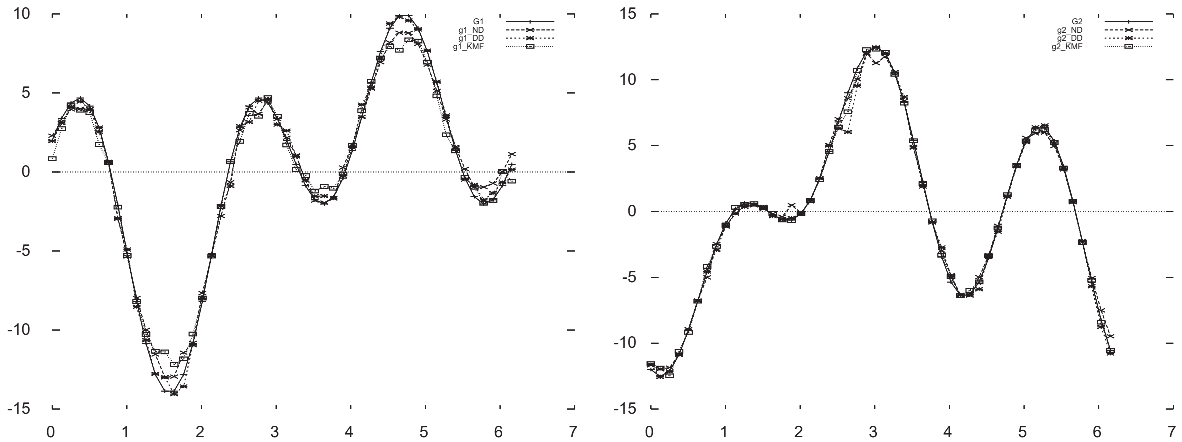


Fig. 1. The domain Ω and the boundaries Γ_a and Γ_i.



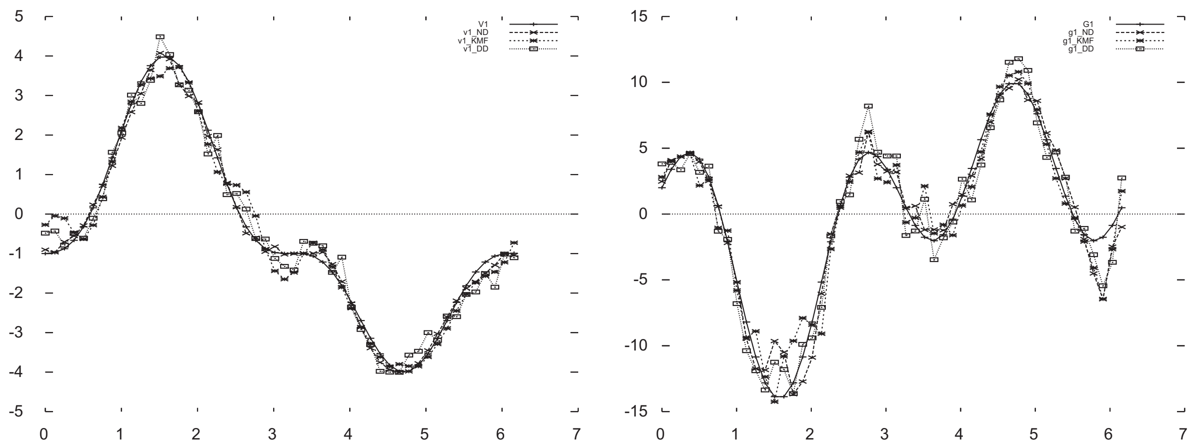
(a) the first component v_1 (b) the second component v_2

Fig. 2. First test: the reconstructed velocity on Γ_i.



(a) the first component g_1 (b) the second component g_2

Fig. 3. First test: the reconstructed stress tensor on Γ_i.



(a) the component v_1 (b) the component g_1

Fig. 4. The reconstructed v_{ND} and stress tensor g_{ND} on Γ_i with 5% of noise.

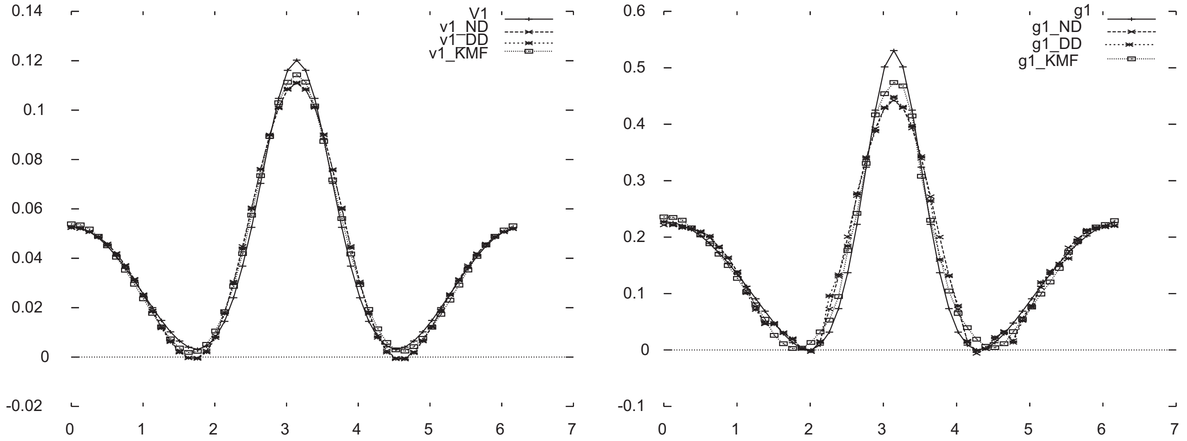


Fig. 5. Singular data: reconstructed velocity and the stress tensor on Γ_i with $a = 0.4$.

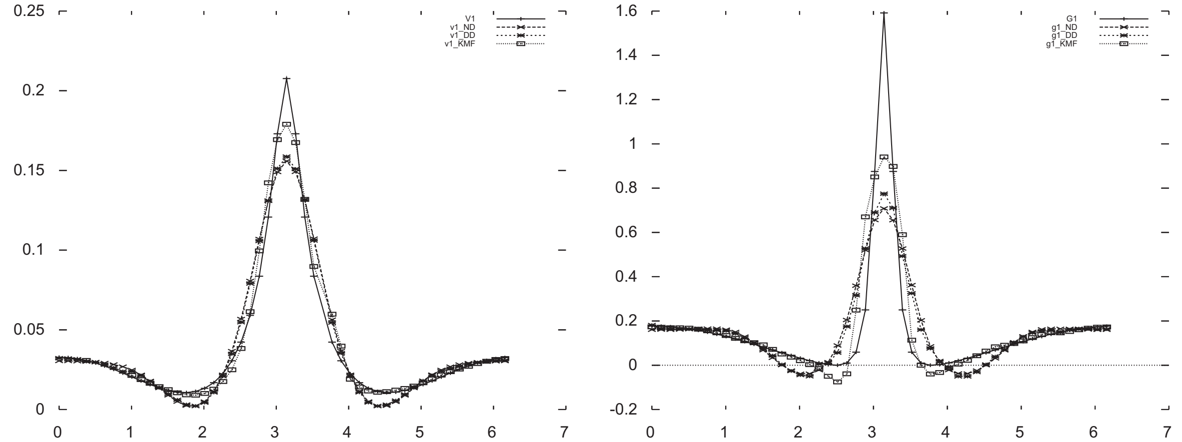


Fig. 6. Singular data: reconstructed velocity and the stress tensor on Γ_i with $a = 0.8$.

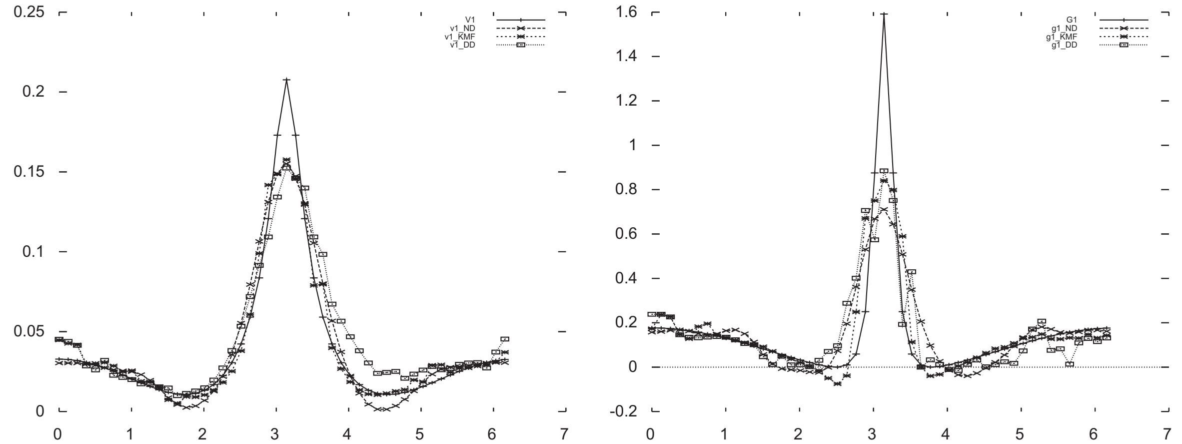


Fig. 7. Singular data: reconstructed velocity and the stress tensor on Γ_i with $a = 0.8$.

Table 2
The number of iterations.

	ND	DD	KMF's method
Number of iterations for polynomial boundary and $\varepsilon = 10^{-3}$	30	35	45
Number of iterations in the presence of singularities with $a = 0.4$ and $\varepsilon = 10^{-5}$	56	55	67
Number of iterations in the presence of singularities with $a = 0.8$ and $\varepsilon = 10^{-5}$	152	217	257

$a = 0.8$. The Fig. 5 shows the reconstructed velocity and the stress tensor for $a = 0.4$ as well as the exact ones. The Fig. 6 shows the reconstructed velocity and the stress tensor for $a = 0.8$ as well as the exact ones. In Fig. 7 the recovered data is for 5% of noise with $a = 0.8$.

The procedure proposed in this paper involves solving an interfacial equation, whose evaluation of the forward problem requiring four computations of a direct solution at each iteration of the Neumann–Dirichlet case, three computations of a direct solution at each iteration of the Dirichlet–Dirichlet case whereas KMF's algorithm needs two computations of a direct solution. The stopping criterion of the KMF relies on the energy. For the examples presented, the numbers of iterations to achieve convergence for each method are shown in Table 2. It is noticed that the KMF's algorithm has difficulties to converge in the presence of singularities, and is generally computationally expensive.

6. Conclusion

In this work, we have investigated the Cauchy problem for the viscous stationary Stokes-system. The unknown boundary data are characterized as the solution of an optimization problem. The minimisation process is achieved through the resolution of the first optimality condition which relies on solving an interfacial equation. Numerical trials highlight the efficiency of the proposed method. The more realistic case of non smooth domains namely domains with corners, is under consideration. Applications to leak detection is also in progress.

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